Analysis of Algorithms

Graph Theory

Algorithmic Questions in Graph Theory.

Non-Graph Theoretic Algorithms

- Sorting
- Searching
- GCD, number theoretic Algorithms

Why Graph Theory

- Mathematica logic
- U - universe of people.

Parent $\subseteq U \times U$ $\qquad$ x is a Parent of y

- not symetric $\qquad$ $x \longrightarrow y$

x Ancestor of y (if $\exists z$, x is a parent of z

$\lor$ (x is a Parent of y) $\land$ (z is an Ancestor of y))

Symetric Relationship

Sibling(x, y)

Automata Theory

Σ is a finite set, called an **ALPHABET**

$\Sigma = \{0, 1\}$

$\Sigma^* = \bigcup_{i \geq 0} \Sigma^i$

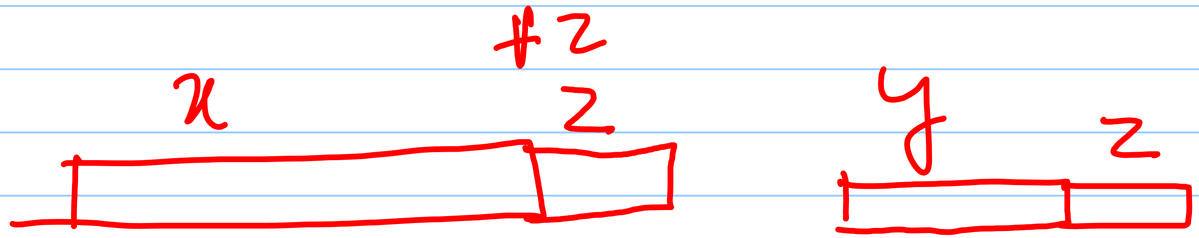
$\Sigma^i = \underbrace{\Sigma \times \Sigma \times \ldots \Sigma}_{i \text{ times}}$

$= \{000\cdot 0, 00\ldots 1, \ldots 111\ldots 1\}$

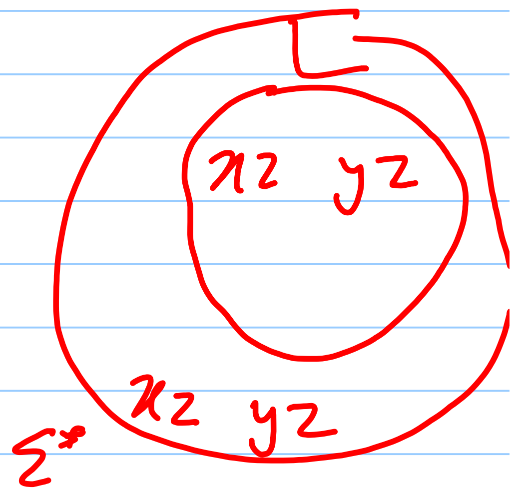
$\Sigma^0 = \{\epsilon\}$

Consider $L \subseteq \Sigma^*$.

$$R_L = \left\{ (x, y) \;\middle|\; \begin{array}{l} x \in \Sigma^*,\ y \in \Sigma^* \\ \underline{\forall}\, z \in \Sigma^* \\ (xz \in L \iff yz \in L) \end{array} \right\}$$



Suffix



1) Symmetric 2) $(x, y)\ (y\, w) \Rightarrow (x\, w)$ Transitive

3) (x, x) – Reflexive.

R_L is a EQUIVALENCE RELATION

partitions Σ^* into equivalence classes

Q1 : Given R_L, How many equivalence classes?

Q2 : If R is a Eq. Relation over a finite set : How many equivalence classes?

Eq. Reln over a finite set

x same country y - # of equivalence classes = # of countries.

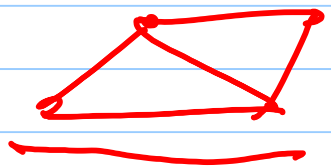
- x $\boxed{\text{connected}}$ y $\Longleftrightarrow$ x same country y

x is an Ancestor of y 	$x \xrightarrow{\text{Parent}} z \rightsquigarrow_{\text{Ancestor}} y$

Path in the graph of Parent Relationship defines Ancestor Relationship.

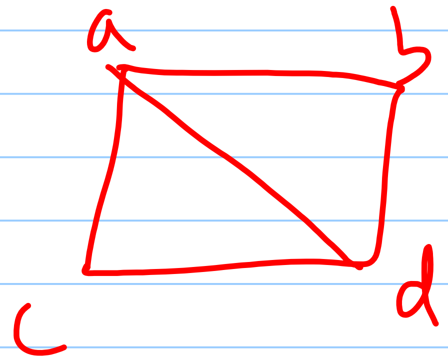
Earliest Graph Theoretic Questions

- Königsberg Bridge problem
- Given valencies ? of different atoms they wanted to know the compounds
- Is a graph connected
- Given a degree sequence, is there a graph with this degree sequence

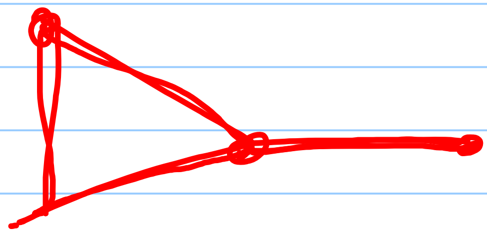


3, 3, 2, 2 ? 3, 3, 2, 2, 1, 1, 1

- How many spanning trees of an undirected graph?



- Does the <u>rook's</u> move graph have a cycle that spans all vertices (HAMILTONIAN)
- <u>Eulerian Tour</u> — Draw G without taking off <u>pen</u> & each edge exactly 1.



Non-Eulerian
Because of an odd-degree vertex.

Scope

Graph Theoretic Problem → Structural Result → Algorithm → Efficient implementation with Data structure

Some Properties of undirected Graphs

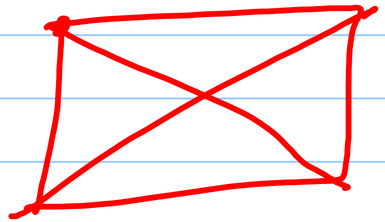
- How many vertices of odd degree?

$$\sum_{i=1}^{n} d_i = 2 \cdot m \qquad \therefore \text{ \# of vertices of odd degree is an even \#}$$

$$\underbrace{\sum d_i}_{\text{even}} + \underbrace{\sum d_i}_{\text{odd}} = \underline{2m}$$

- $d_1 \geq d_2 \geq d_3 \cdots \geq d_n$ is <u>GRAPHIC</u>

1) $\sum d_i$ is even.

2)  $d_i \leq n-1$ for each i

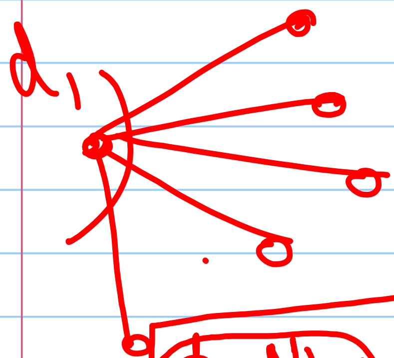
3) $\underline{d_1 \ldots d_{n-1}}$ must be $\underline{\text{graphic}}$

$d_1 \geq \ldots \geq d_n$ is graphic $\iff$ $d_1 \ldots d_{n-1}$ is graphic.

$\underline{n-1 \geq n-1, \ldots \geq n-1}$

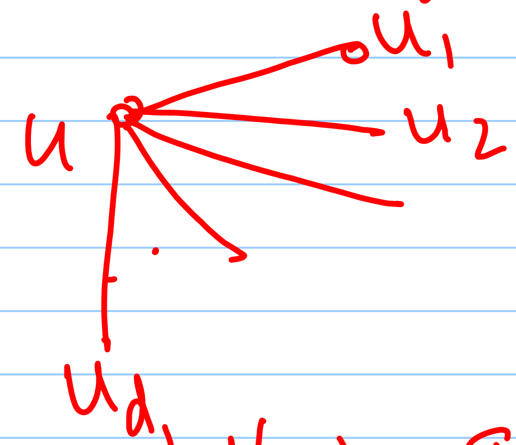
$d_1 \geq d_2 \ . \ \geq d_n$ is graphic

$d_2 - 1, d_3 - 1, \ldots d_{d_1+1} - 1, \ldots d_n$ "is" graphic.



There is a graph realizing $d_1 \geq \ldots d_n$ in which a vertex of degree d_1 has d_1 Nbrs of degree $d_2 \ldots d_{d_1+1}$

- Let $\underline{G}$ be a graph realizing $\underline{d_1 \geq d_2 \cdots \geq d_n}$
- Let u be a vertex degree d_1
- Consider $\underline{N(u)}$



Consider the degrees in $\underline{N(u)}$ $\underline{g_1 \geq \cdots \geq g_{d_1}}$

<u>Case 1</u> • $d_2 \geq d_3 \cdots \geq d_{d_1+1}$ same as $g_1 \geq \cdots \geq g_{d_1}$, our claim is correct

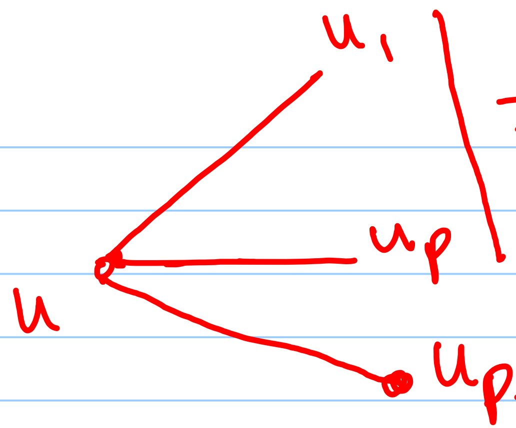
Case 2 $d_2 \geq \cdots \geq d_{d_1+1}$ are different

$g_1 \geq \cdots \geq g_{d_1}$

Let p be the maximal prefix of the 2 non-increasing sequences which are identical.

$\underline{d_2 \geq \cdots \geq d_{p+1}} \mid d_{p+2} \cdots d_{d_1+1} \leftarrow$ Seq

$\underline{g_1 \geq \cdots \geq g_p} \mid g_{p+1} \quad g_{d_1} \leftarrow$ graph



$\#$ is $< d_1$

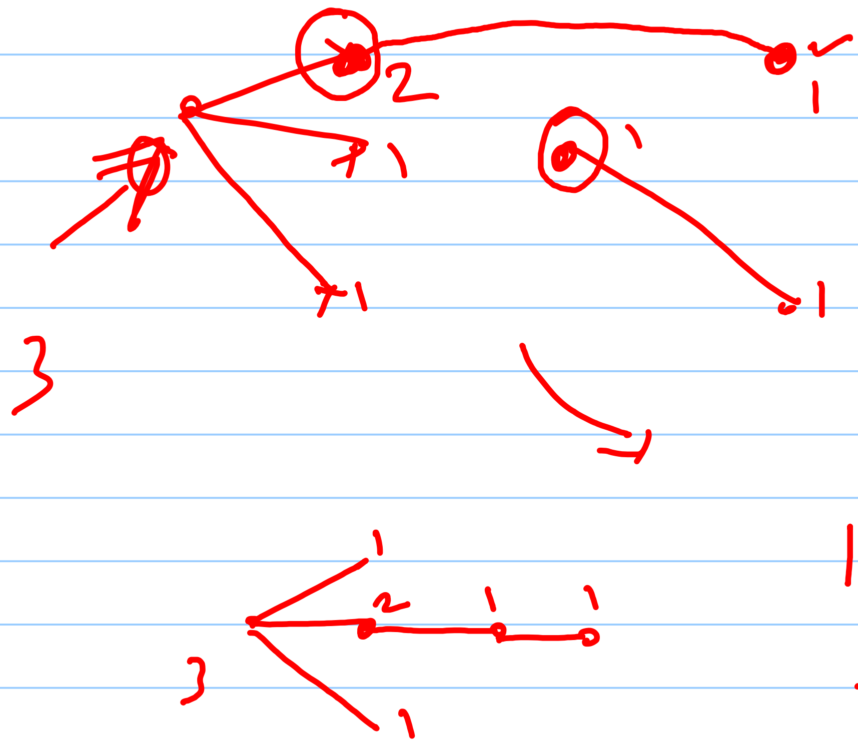
$g_1 = d_2$
$g_2 = d_3$
$g_p = d_{p+1}$

u_{p+1} has degree $\boxed{g_{p+1} < d_{p+2}}$

If $g_{p+1} > d_{p+2}$ then it must be in $\{g_1 \ldots g_p\}$

$d_1 \geq d_2 \geq \ldots d_p \geq d_{p+1} \geq d_{p+2}$

3 2 1 1 1 1 1 → d



G → Only realization with 2 conn. components.

$10 = \sum d_i = 2 \cdot |E|$

$\therefore |E| = 5$

# of vertices = 7

$d_2 \cdots \geq d_{p+1} \geq d_{p+2}$

$g_1 \cdot \geq g_p \qquad \overset{\#}{g_{p+1}}$

$\therefore \boxed{g_{p+1} < d_{p+2}}$

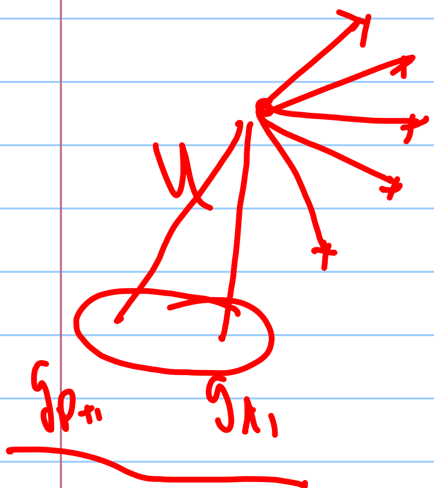
$g_{p+2} < d_{p+1}$

$g_{d_1} < d_{p+2}$

$$\begin{array}{ccccccc} d_1 & \geq & d_2 & \cdots & \geq d_{p+1} & \geq & d_{p+2} \\ & & \shortparallel & & \shortparallel & & \# \\ g & & g_1 & & g_p & & \{g_{p+1} \cdots g_{d_1}\} \end{array}$$

Input $d_1 \geq d_2 \geq \cdots d_{p+1} \geq d_{p+2} \quad d_n$

$\qquad\qquad g_1 \qquad\qquad g_p$



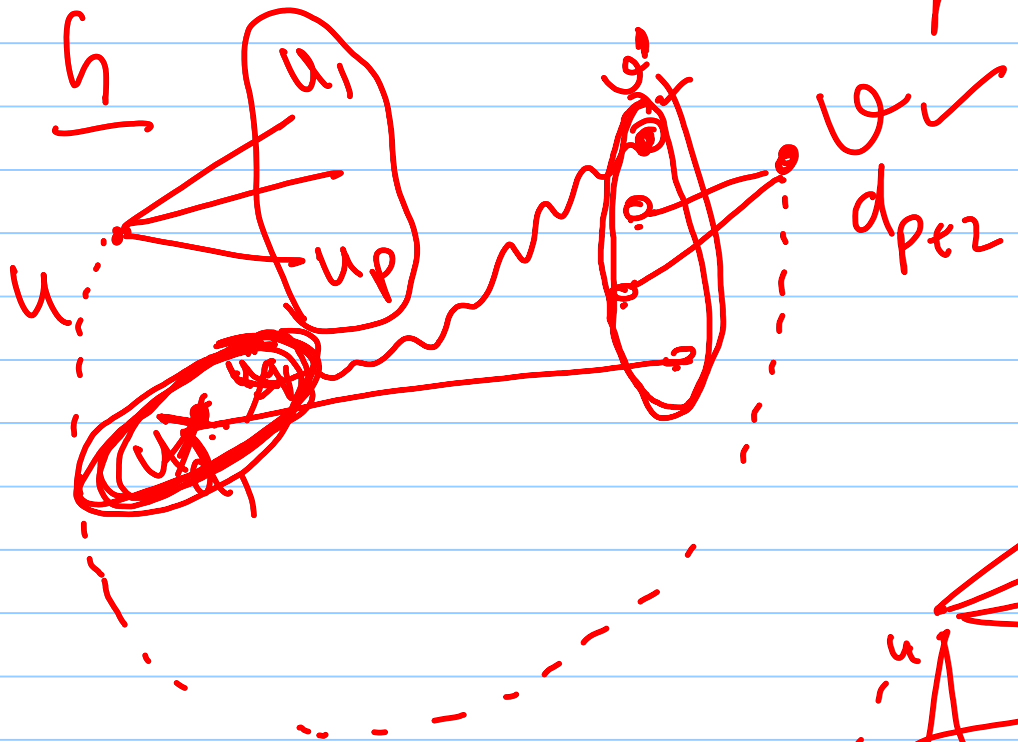
$\{g_{p+1} \cdots g_{d_1}\}$ occurs

after $d_2 \cdots d_{p+1}$ and is

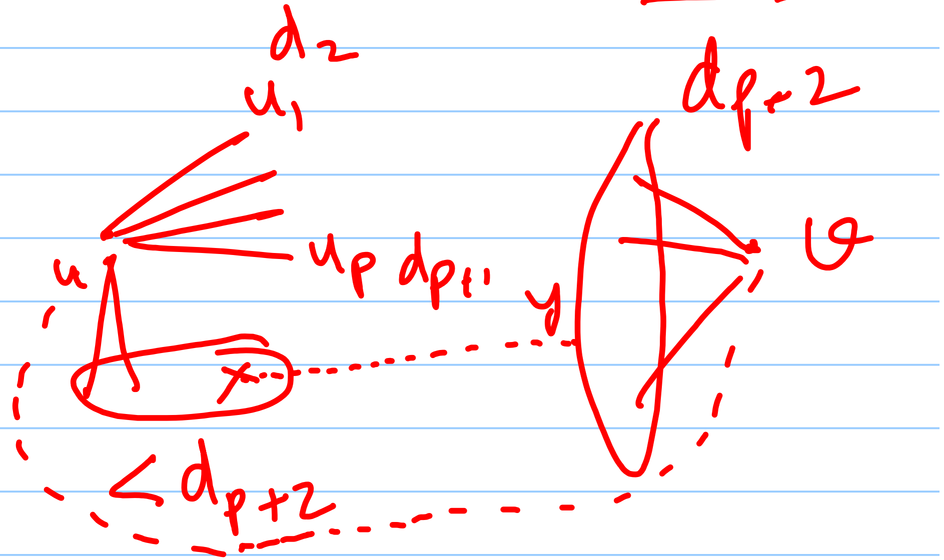
Not equal to d_{p+2}.

$\therefore$ they are smaller than $\boxed{d_{p+2}}$

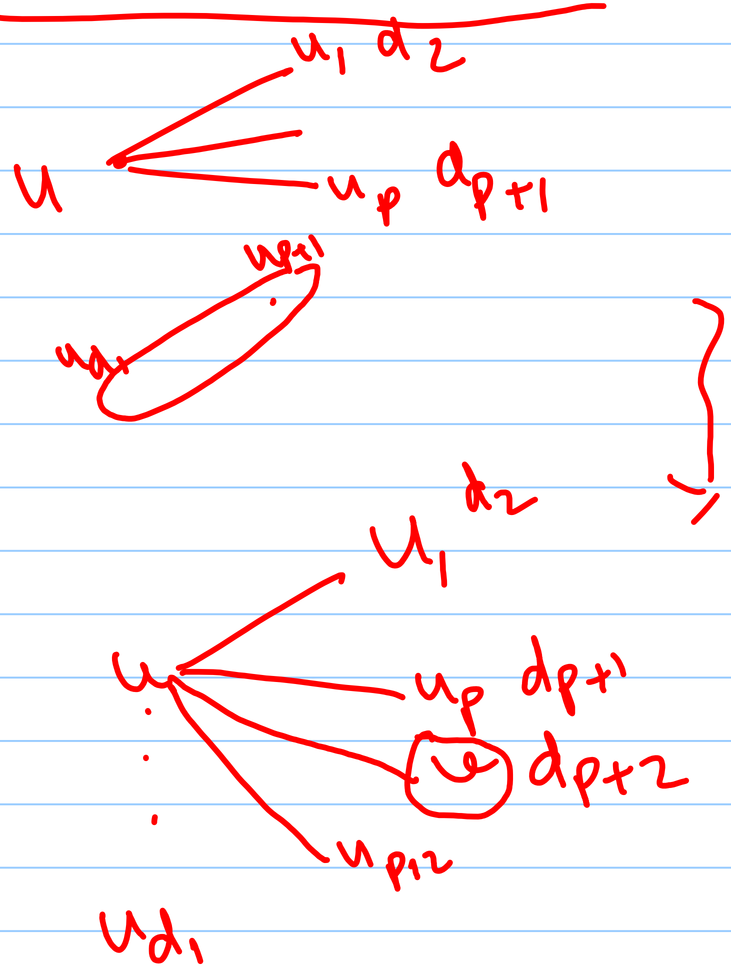
There is a vertex of degree d_{p+2} in $\underline{G}$



$d_2 \cdots d_{p+1}\ \underline{d_{p+2}}$



Havel-Hakimi Thm



d_{p+2}

• v

$d_1 \geq \dots \geq d_n \geq 0$

graphic

$\iff$

$d_2 - 1, \dots d_{d_1+1} - 1, \dots d_n \geq 0$

is graphic.

$\Rightarrow$ interesting ✓

$\Leftarrow$ Easy

RecursiveHH(n, $D[1..n]$)

$\Sigma n \log n$

$n^2 \log n$

1) If $n = 0$, return yes

2) If any degree is $-ve$ return no

3) Remove d_1, reduce n

consider $d_2 - 1, \ldots d_{d_1+1} - 1, \ldots d_n$

Sort

RecursiveHH($n-1$, $D[1..n-1]$)